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SOME ASYMPTOTIC PROPERTIES OF MACROPARAMETERS OF RAREFIED GAS EXPANSION INTO VACUUM

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The problem of expansion of a plane layer of a Maxwellian gas into vacuum is considered. The expansion in terms of the Knudsen numbers has a logarithmic singularity at large values of time, and this singularity can be removed by slight stretching of the time coordinate. An asymptotic solution is derived describing a distant field of flow in a plane stationary stream for the Knudsen numbers tending to zero.

Investigation of the plane expansion of a rarefied gas of finite mass into vacuum at small initial Knudsen numbers [1], shows that the continuous medium regime is violated when the values of time are of the order of $\mathrm{Kn}^{-\sigma}$, $\sigma = {}^3/_2 (1 - \nu)^{-1}$, under the assumption that the viscosity and temperature are connected by the power relation $\mu = T^\nu$. In a particular case when $\nu = 1$ is studied below, a uniformly valid solution can be obtained using the Lighthill method of deformed coordinates. This method was applied for the first time in [2] to a steady flow of a rarefied gas into vacuum.

1. Before studying the expansion of a Maxwellian gas, we shall give an asymptotic solution describing a distant flow field in a plane stationary stream, since the equations in the outer region remain the same on passing to a one-dimensional unsteady problem.

A steady expansion of gas with cylindrical symmetry was analyzed in [2—4]. We shall consider a more general flow in which the macroparameters depend not only on the radius, but also on the azimuthal angle φ. The present formulation can serve as a model for the conditions which exist when a rarefied gas flows through a plane slit. Let us write the generalized kinetic Krook's equation in cylindrical coordinates

$$\xi_r \frac{\partial f}{\partial r} + \frac{\xi_\varphi}{r}\frac{\partial f}{\partial \varphi} + \frac{\xi_\varphi^2}{r}\frac{\partial f}{\partial \xi_r} - \frac{\xi_r \xi_\varphi}{r}\frac{\partial f}{\partial \xi_\varphi} = AnT^{1-\nu}(f^+ - f) \qquad (1.1)$$

$$A \sim \mathrm{Kn}^{-1}, \quad \mu = T^\nu$$

All quantities are reduced to their dimensionless form relative to the gas parameters near the slit; the Knudsen number defined across the slit width is assumed small; the inviscid flow equations become valid when $\mathrm{Kn} \to 0$. The radial character of the distant flow enables us to write the gas dynamic solution in the form

$$u = w_0 + \frac{w_1(\varphi)}{r^{2/3}} + \ldots, \quad n = \frac{\rho_0(\varphi)}{r} + \frac{\rho_1(\varphi)}{r^{5/3}} + \ldots \qquad (1.2)$$

$$v = \frac{v_1(\varphi)}{r^{2/3}} + \ldots, \quad T = \frac{[\rho_0(\varphi)]^{2/3}}{r^{2/3}} + \frac{q_1(\varphi)}{r^{4/3}} + \ldots, \quad r \to \infty$$

Here u and v denote, respectively, the radial and transversal velocity components, w_0 is the limiting velocity during the isentropic flow into vacuum and r is the coordinate defining the distances from the z-axis, along which the gas parameters are not changed. By specifying $\rho_0(\varphi)$, we define the remaining functions w_1, v_1, ρ_1, q_1. The form of $\rho_0(\varphi)$ follows from the exact solution at $r = O(1)$. Substituting the expressions (1.2) into (1.1) and equating the convective and the collision terms, we find that the inner gas dynamic solution is ineffective at the distances

$$r = O(A^\sigma), \quad \sigma = {}^3/_2(1-\nu)^{-1}$$

Following a method used earlier in investigating an axisymmetric flow [6], we pass to the variables $r_1 = rA^{-\sigma}, \quad n_1 = nA^\sigma, \quad T_1 = TA^{2\sigma/3}$

$$\alpha = (\xi_r - u)A^{\sigma/3}, \quad \beta = (\xi_\varphi - v)A^{\sigma/3}, \quad \gamma = \xi_z A^{\sigma/3}$$

Equation (1.1) now becomes

$$ur_1\frac{\partial f}{\partial r_1} - \alpha r_1\frac{\partial f}{\partial \alpha}\frac{\partial u}{\partial r_1} - \alpha r_1\frac{\partial f}{\partial \beta}\frac{\partial v}{\partial r_1} + v\frac{\partial f}{\partial \varphi} - \beta\frac{\partial f}{\partial \alpha}\frac{\partial u}{\partial \varphi} - \beta\frac{\partial f}{\partial \beta}\frac{\partial v}{\partial \varphi} + \tag{1.3}$$
$$2\beta v\frac{\partial f}{\partial \alpha} - (\beta u + \alpha v)\frac{\partial f}{\partial \beta} - A^{\sigma/3}ur_1\frac{\partial f}{\partial \alpha}\frac{\partial u}{\partial r_1} + ur_1\frac{\partial f}{\partial \beta}\frac{\partial v}{\partial r_1} + v\frac{\partial f}{\partial \alpha}\frac{\partial u}{\partial \varphi} +$$
$$v\frac{\partial f}{\partial \beta}\frac{\partial v}{\partial \varphi} - v^2\frac{\partial f}{\partial \alpha} + uv\frac{\partial f}{\partial \beta}\Big) + A^{-\sigma/3}\left(r_1\alpha\frac{\partial f}{\partial r_1} + \beta\frac{\partial f}{\partial \varphi} + \beta^2\frac{\partial f}{\partial \alpha} - \alpha\beta\frac{\partial f}{\partial \beta}\right) =$$
$$r_1 n_1 T_1^{1-\nu}(f^+ - f)$$

Integrating (1.3) over the space of velocities with weights α and β, we obtain the following equations for the macroscopic velocity:

$$ur_1\frac{\partial u}{\partial r_1} + v\frac{\partial u}{\partial \varphi} - v^2 = O(A^{-2\sigma/3}) \tag{1.4}$$
$$ur_1\frac{\partial v}{\partial r_1} + v\frac{\partial v}{\partial \varphi} + uv = O(A^{-2\sigma/3})$$

The outer solution admits the representation

$$u = U_0 + U_1A^{-2\sigma/3} + \ldots, \quad n_1 = N_0 + N_1A^{-2\sigma/3} + \ldots$$
$$v = V_0 + V_1A^{-2\sigma/3} + \ldots, \quad f = F_0 + F_1A^{-\sigma/3} + \ldots$$

From Eqs. (1.4) and the equation of continuity we obtain, utilizing the conditions of matching to the inner solution (1.2)

$$U_0 = w_0, \quad V_0 = 0, \quad N_0 = \rho_0(\varphi)/r_1$$

Taking into account the results obtained, we can write a kinetic equation for the function F_0 in the form

$$r_1U_0\frac{\partial F_0}{\partial r_1} - \beta U_0\frac{\partial F_0}{\partial \beta} = \rho_0(\varphi)\tau^{1-\nu}(F^+ - F) \tag{1.5}$$

Let us introduce the quantities

$$\overline{\alpha^2} = \frac{1}{N_0r_1}\int \alpha^2F_0\,d\alpha\,dB\,d\gamma, \quad \overline{B^2} = \frac{1}{N_0r_1^3}\int B^2F_0\,d\alpha\,dB\,d\gamma \tag{1.6}$$
$$\overline{\gamma^2} = \frac{1}{N_0r_1}\int \gamma^2F_0\,d\alpha\,dB\,d\gamma, \quad B = \beta r_1, \quad \frac{3}{2}\tau = \overline{\alpha^2} + \overline{B^2} + \overline{\gamma^2}$$

From (1.5) we obtain the following moment equations:

$$\frac{\partial}{\partial r_1}\overline{\alpha^2} = \frac{N_0}{U_0}\tau^{1-\nu}\left(\frac{\tau}{2} - \overline{\alpha^2}\right), \quad \frac{\partial}{\partial r_1}\overline{\gamma^2} = \frac{N_0}{U_0}\tau^{1-\nu}\left(\frac{\tau}{2} - \overline{\gamma^2}\right) \tag{1.7}$$

$$\frac{\partial}{\partial r_1} r_1^2 \overline{B^2} = \frac{N_0}{U_0} \tau^{1-\nu} r_1^2 \left(\frac{\tau}{2} - \overline{B^2}\right)$$

Adding Eqs. (1.7) together and using (1.6), we obtain

$$\frac{3}{4}\frac{\partial \tau}{\partial r_1} = -\frac{\overline{B^2}}{r_1} \qquad (1.8)$$

Substitution of (1.8) into the last equation of the system (1.7) now yields the following equation for the temperature:

$$r_1^2 \frac{\partial^2 \tau}{\partial r_1^2} + r_1 \frac{\partial \tau}{\partial r_1}(3 + \rho(\varphi)\tau^{1-\nu}) + \frac{2}{3}\rho(\varphi)\tau^{2-\nu} = 0 \qquad (1.9)$$

$$\rho(\varphi) = \rho_0(\varphi) \,/\, U_0$$

Using the transformation

$$\theta = \ln t, \quad \partial\tau \,/\, \partial\theta = -2\tau\Psi$$

we reduce (1.9) to the following first order equation:

$$\frac{\partial \Psi}{\partial \tau} = \frac{1}{\tau}(1 - \Psi) + \frac{\rho}{2\tau^{\nu}}\left(1 - \frac{1}{3\Psi}\right) \qquad (1.10)$$

The singularity $\tau = 0$, $\Psi = 0$ is a node, and the point $\tau = 0$, $\Psi = 1$ is a saddle. The possibility of matching with the inner solution is determined by the behavior of the integral curves at the point of infinity. A unique integral curve exists which emerges from the saddle-type singularity $\tau = \infty$, $\Psi = {}^1/_3$, and the corresponding solution in the physical plane is

$$\tau = G(\varphi) \,/\, r_1^{2/3}, \quad r_1 \to 0$$

Comparing this with (1.2), we find the multiplying factor $G(\varphi) = [\rho_0(\varphi)]^{2/3}$. The integral curve emerging from the singularity $\tau = \infty$, $\Psi = {}^1/_3$, cannot arrive at the saddle point $\tau = 0$, $\Psi = 1$, since by (1.10) we have $\partial\Psi / \partial\tau > 0$ when ${}^1/_3 < \Psi < 1$. Thus the asymptotics with $r_1 \to \infty$ is determined by the solution entering the node $\tau = 0$, $\Psi = 0$

$$\Psi = \frac{\rho}{6}\tau^{1-\nu} + O(\tau^{2-2\nu}) \qquad (1.11)$$

and this yields the following expression for the temperature:

$$\tau = \left[\left(\frac{1}{3}\frac{\rho_0(\varphi)}{U_0}\ln r_1\right)(1-\nu)\right]^{-1/(1-\nu)}, \quad r_1 \to \infty \qquad (1.12)$$

The properties of the solution shown above can be confirmed quantitatively by the results of the numerical investigation in [7] of the flow from a plane slit, according to which the density and velocity are not significantly affected by the decrease in the Knudsen number, while the temperature at the distance from the slit shows a tendency to decrease practically to zero. Moreover, the increase in temperature with increasing azimuthal angle φ noted in [7] follows from the formula (1.12) since the function $\rho_0(\varphi)$ decreases on moving away from the central streamline.

2. Next we consider a one-dimensional unsteady expansion of gas into vacuum, using the conservation equations

$$\frac{\partial n}{\partial t} + \frac{\partial nu}{\partial x} = 0 \qquad (2.1)$$

$$n\frac{\partial u}{\partial t} + nu\frac{\partial u}{\partial x} = -\frac{1}{2}\frac{\partial nT}{\partial x} + \frac{4}{3}\frac{\partial}{\partial x}\left(\bar{\mu}\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial x}P_{xx}^{(2)}$$

$$\frac{3}{2} n \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x}\right) = -nT \frac{\partial u}{\partial x} + \frac{8}{3} \bar{\mu} \left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial}{\partial x} k \frac{\partial T}{\partial x} -$$
$$2P_{xx}^{(2)} \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} q_x^{(2)}$$
$$\bar{\mu} = \frac{1}{2} \mu A^{-1}, \quad k = \frac{15}{8} \mu A^{-1}, \quad \mu = T^{\nu}, \quad A = \frac{8}{5\sqrt{\pi}} \frac{1}{\text{Kn}} \to \infty$$

where we use the dimensionless coordinates of [1]. By $P_{xx}^{(2)}$, $q_x^{(2)}$ we denote the Barnett stress and heat flux components which, for the generalized Krook equations, are written in the form

$$P_{xx}^{(2)} = A^{-2} \frac{\mu^2}{nT} \left[\frac{8}{9} \left(\frac{\partial u}{\partial x}\right)^2 \left(1 - \frac{1}{2} \frac{T}{\mu} \frac{d\mu}{dT}\right) + \frac{1}{6} \frac{\partial^2 T}{\partial x^2} + \frac{1}{2} \left(\frac{\partial T}{\partial x}\right)^2 \frac{1}{\mu} \frac{d\mu}{dT} -\right.$$
$$\left.\frac{1}{3} \frac{\partial}{\partial x} \left(\frac{T}{n} \frac{\partial n}{\partial x}\right)\right], \quad q_x^{(2)} = A^{-2} \frac{\mu^2}{nT} \left[\frac{63}{8} \frac{\partial u}{\partial x} \frac{\partial T}{\partial x} - \frac{T}{n} \frac{\partial u}{\partial x} \frac{\partial n}{\partial x} -\right.$$
$$\left.\frac{7}{8} T \frac{\partial^2 u}{\partial x^2} - \frac{7}{8} \frac{T}{\mu} \frac{\partial u}{\partial x} \frac{\partial T}{\partial x} \frac{d\mu}{dT}\right]$$

Substituting into Eqs. (2.1) the expansions

$$q = q_0 + A^{-1} q_1 + A^{-2} q_2 + \dots \quad (q = n, u, T) \tag{2.2}$$

we obtain the Euler equations in the zero approximation. For the large values of time we write the solution of the Euler equations in the form

$$n_0 = \frac{g_0(\lambda)}{t} + \frac{g_1(\lambda)}{t^{5/3}} + \dots, \quad u_0 = \lambda + \frac{\omega(\lambda)}{t^{2/3}} + \dots \tag{2.3}$$
$$T_0 = \frac{c_0(\lambda)}{t^{2/3}} + \frac{c_1(\lambda)}{t^{4/3}} + \dots, \quad \lambda = \frac{x}{t}$$

The functions g_0, g_1, ω, c_0, c_1 are connected by the equations

$$g_1 = -\frac{15}{4} g_0^{4/3} \left(\frac{g_0'}{g_0^{2/3}}\right)', \quad \omega = -\frac{5}{2} \frac{g_0'}{g_0^{1/3}}$$
$$c_0 = g_0^{2/3}, \quad c_1 = \frac{2}{3} \frac{g_1}{g_0^{1/3}}$$

where the prime denotes differentiation with respect to λ. The exact form of the function $g_0(\lambda)$ depends on the initial conditions. In the next approximation we have

$$\frac{\partial n_1}{\partial t} + \frac{\partial}{\partial x} n_1 u_0 + \frac{\partial}{\partial x} n_0 u_1 = 0 \tag{2.4}$$
$$n_1 \frac{\partial u_0}{\partial t} + n_0 \frac{\partial u_1}{\partial t} + n_0 u_0 \frac{\partial u_1}{\partial x} + n_1 u_0 \frac{\partial u_0}{\partial x} + n_0 u_1 \frac{\partial u_0}{\partial x} = -\frac{1}{2} \frac{\partial}{\partial x} n_1 T_0 -$$
$$\frac{1}{2} \frac{\partial}{\partial x} n_0 T_1 + \frac{2}{3} \frac{\partial}{\partial x} \left(T_0^{\nu} \frac{\partial u_0}{\partial x}\right)$$
$$\frac{3}{2} \left(n_1 \frac{\partial T_0}{\partial t} + n_0 \frac{\partial T_1}{\partial t} + n_1 u_0 \frac{\partial T_0}{\partial x} + n_0 u_1 \frac{\partial T_0}{\partial x} + n_0 u_0 \frac{\partial T_1}{\partial x}\right) =$$
$$-n_1 T_0 \frac{\partial u_0}{\partial x} - n_0 T_1 \frac{\partial u_0}{\partial x} - n_0 T_0 \frac{\partial u_1}{\partial x} + \frac{4}{3} T_0^{\nu} \left(\frac{\partial u_0}{\partial x}\right)^2 + \frac{15}{8} \frac{\partial}{\partial x} \left(T_0^{\nu} \frac{\partial T_0}{\partial x}\right)$$

For $\nu \neq 1$ Eqs. (2.4) and the similar system for n_2, u_2 and T_2 admit the asymptotics

$$T_1 \sim t^{-2/3+\varepsilon}, \quad u_1 \sim t^{-2/3+\varepsilon}, \quad n_1 \sim t^{-5/3+\varepsilon}$$
$$T_2 \sim t^{-2/3+2\varepsilon}, \quad u_2 \sim t^{-4/3+2\varepsilon}, \quad n_2 \sim t^{-7/3+2\varepsilon}, \quad \varepsilon = {}^2/_3 (1 - \nu)$$

Consequently if $\nu > 1$, T_1 and T_2 decrease faster than T_0 as $t \to \infty$, and the inner gas dynamic solution is uniformly suitable. If $\nu < 1$, the expansion (2.2) diverges as $t \to \infty$ due to the increasing higher order approximations. The latter case was studied in [1]. In the region $t_1 = A^{-\sigma} t = 0\,(1)$ the temperature satisfies an equation of the type (1.9)

$$t_1^2 \frac{\partial^2 \tau}{\partial t_1^2} + t_1 \frac{\partial \tau}{\partial t_1}(3 + g_0(\lambda)\,\tau^{1-\nu}) + \frac{2}{3}\, g_0(\lambda)\,\tau^{2-\nu} = 0, \quad \tau = TA^{1/(1-\nu)} \tag{2.5}$$

For $\nu = 1$ Eq. (2.5) becomes linear and has the solution

$$\tau = G_1 / t_1^{\delta_1} + G_2 / t_2^{\delta_2}\, \delta_{1,2} = 1 + \frac{g_0}{2} \pm \sqrt{1 + \frac{g_0}{3} + \frac{g_0^2}{4}} \tag{2.6}$$

We see from (2.6) that the condition of matching with the inner solution $T = c_0 / t^{2/3}$ is violated. Since at $\nu = 1$ the introduction of the variable t_1 no longer makes sense, we note that an equation of the type (2.5) follows directly from the kinetic equation under the assumption that $u = \lambda$, $t \to \infty$. Moreover, the function $g_0(\lambda)$ must be replaced by $A g_0(\lambda)$, $A \to \infty$. Then, assuming that $T = T_0 + A^{-1}T_1 + A^{-2}T_2 + \ldots$, we obtain

$$T_0 = \frac{c_0(\lambda)}{t^{2/3}}, \quad \frac{\partial T_1}{\partial t} + \frac{2}{3}\frac{T_1}{t} = \frac{8}{9}\frac{c_0}{g_0 t^{5/3}}, \quad \frac{\partial T_2}{\partial t} + \frac{2}{3}\frac{T_2}{t} = \frac{8}{9}\frac{T_1}{g_0 t} - \frac{8}{27}\frac{c_0}{g_0^2 t^{8/3}}$$

i.e. the Navier-Stokes and Barnett approximations contain a logarithmic singularity at infinity

$$T_1 = \frac{c_1}{t^{2/3}} + \frac{8}{9}\frac{c_0}{g_0}\frac{\ln t}{t^{2/3}}, \quad T_2 = \frac{c_2}{t^{2/3}} + \frac{8}{9}\frac{1}{g_0}\left(c_1 - \frac{c_0}{3g_0}\right)\frac{\ln t}{t^{2/3}} + \frac{64}{72}\frac{c_0}{g_0^2}\frac{\ln^2 t}{t^{2/3}}$$

Naturally, the same results are also obtained from the conservation equation. The appearance of nonuniformity is connected with the term $4/3\, I_0 (\partial u_0/\partial x)^2$ in the equation for T_1 of the system (2.4), and with the terms

$$\frac{4}{3} T_1 \left(\frac{\partial u_0}{\partial x}\right)^2, \quad \frac{8}{9}\frac{T_0}{n_0}\left(\frac{\partial u_0}{\partial x}\right)^3 \left[1 - \frac{1}{2}\frac{T_0}{\mu_0}\frac{d\mu_0}{dT_0}\right]$$

in the equation for T_2.

To remove the logarithmic singularity, we impose a small perturbation on the independent coordinates

$$t = \zeta + A^{-1}\Theta_1(\zeta, \eta) + A^{-2}\Theta_2(\zeta, \eta) + \ldots \tag{2.7}$$
$$x = \eta + A^{-1}X_1(\zeta, \eta) + A^{-2}X_2(\zeta, \eta) + \ldots$$

The macroscopic quantities are expanded into the series

$$n(t, x, A) = N_0(\zeta, \eta) + A^{-1}N_1(\zeta, \eta) + \ldots \tag{2.8}$$
$$u(t, x, A) = V_0(\zeta, \eta) + A^{-1}V_1(\zeta, \eta) + \ldots$$
$$T(t, x, A) = \tau_0(\zeta, \eta) + A^{-1}\tau_1(\zeta, \eta) + \ldots$$

The functions Θ_i and X_i are determined by the requirement that the singularity does not increase in each subsequent approximation. The transformation (2.7), (2.8) yields the following expressions for (2.1) in the zero approximation

$$\frac{\partial N_0}{\partial \zeta} + \frac{\partial V_0 N_0}{\partial \eta} = 0 \tag{2.9}$$
$$\frac{\partial V_0}{\partial \zeta} + V_0 \frac{\partial V_0}{\partial \eta} = -\frac{1}{2}\frac{\tau_0}{N_0}\frac{\partial N_0}{\partial \eta} - \frac{1}{2}\frac{\partial \tau_0}{\partial \eta}$$
$$\frac{\partial \tau_0}{\partial \zeta} + V_0 \frac{\partial \tau_0}{\partial \eta} = -\frac{2}{3}\tau_0 \frac{\partial V_0}{\partial \eta}$$

In the next higher approximation the last equation of (2. 1) yields

$$\frac{3}{2}\left(N_1\frac{\partial\tau_0}{\partial\zeta}+N_0\frac{\partial\tau_1}{\partial\zeta}+V_1N_0\frac{\partial\tau_0}{\partial\eta}+V_0N_1\frac{\partial\tau_0}{\partial\eta}+N_0V_0\frac{\partial\tau_1}{\partial\eta}\right)=$$
$$-N_1\tau_0\frac{\partial V_0}{\partial\eta}-N_0\tau_1\frac{\partial V_0}{\partial\eta}-N_0\tau_0\frac{\partial V_1}{\partial\eta}+\left[\frac{3}{2}N_0\frac{\partial\tau_0}{\partial\zeta}\frac{\partial\Theta_1}{\partial\zeta}+\right.$$
$$\frac{3}{2}N_0\frac{\partial\tau_0}{\partial\eta}\frac{\partial X_1}{\partial\zeta}+\frac{3}{2}V_0N_0\frac{\partial\tau_0}{\partial\eta}\frac{\partial X_1}{\partial\eta}+\frac{3}{2}V_0N_0\frac{\partial\tau_0}{\partial\zeta}\frac{\partial\Theta_1}{\partial\eta}+$$
$$\left.N_0\tau_0\frac{\partial V_0}{\partial\eta}\frac{\partial X_1}{\partial\eta}+N_0\tau_0\frac{\partial V_0}{\partial\zeta}\frac{\partial\Theta_1}{\partial\eta}\right]+\frac{4}{3}\tau_0\left(\frac{\partial V_0}{\partial\eta}\right)^2+\frac{15}{8}\frac{\partial}{\partial\eta}\left(\tau_0\frac{\partial\tau_0}{\partial\eta}\right)$$

The singularity at infinity is stipulated by the term $4/3\tau_0\,(\partial V_0\,/\,\partial\eta)^2$. Let us set $X_1=0$. The choice of the function Θ_1 from the relation

$$\frac{3}{2}N_0\frac{\partial\Theta_1}{\partial\zeta}\frac{\partial\tau_0}{\partial\zeta}+\frac{3}{2}N_0V_0\frac{\partial\tau_0}{\partial\zeta}\frac{\partial\Theta_1}{\partial\eta}+N_0\tau_0\frac{\partial V_0}{\partial\zeta}\frac{\partial\Theta_1}{\partial\eta}+\frac{4}{3}\tau_0\left(\frac{\partial V_0}{\partial\eta}\right)^2=0 \quad (2.10)$$

eliminates the nonuniform term.

We shall seek the solution of the partial differential equation (2. 10) for the case in which we choose the exact gas dynamic solution [8] to represent the terms N_0, V_0, τ_0, satisfying the Euler equations (2. 9) in the ζ, η variables. The following formulas hold for this gas dynamic solution at large ζ :

$$N_0=\frac{[1-s^2]^{3/2}}{k\zeta},\quad V_0=\frac{\eta}{\zeta},\quad \tau_0=\frac{1-s^2}{k^{2/3}\zeta^{2/3}},\quad k=3\sqrt{\frac{5}{6}},\quad s=\frac{\eta}{k\zeta}$$

In this case the characteristic equation is written in the form

$$\frac{d\eta}{d\zeta}=V_0+\frac{{}^2/_3\tau_0\partial V_0/\partial\zeta}{\partial\tau_0/\partial\zeta}\frac{\eta}{\zeta}\frac{2-5s^2}{1-4s^2} \quad (2.11)$$

Integrating (2. 11) we obtain

$$\zeta=cs\,(1-s^2)^{3/2},\quad c=\text{const} \quad (2.12)$$

The relation (2. 10) is rewritten along the direction of (2. 12) as follows:

$$\frac{d\Theta_1}{d\zeta}=\frac{4k}{3\,[1-s^2]^{1/2}\,[1-4s^2]} \quad (2.13)$$

Integrating (2. 13) under the condition that $\Theta_1\to 0,\ \zeta\to 0$, we obtain

$$\Theta_1=\frac{4k}{3}\zeta\left[1-\left(\frac{\eta}{k\zeta}\right)^2\right]^{-3/2} \quad (2.14)$$

The gas dynamic solution in which the time coordinate t is replaced by ζ is, in accordance with (2. 7), (2. 14), uniformly suitable in the sense that no singularity appears in the terms of the order of A^{-1}. Singularities appearing in the higher order approximations are removed by suitable choice of Θ_i, X_i.

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